LIPSCHITZ HOMOTOPY GROUPS OF THE HEISENBERG GROUPS

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Abstract. Lipschitz and horizontal maps from an n-dimensional space into the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ are abundant, while maps from higher-dimensional spaces are much more restricted. DeJarnette-Hajłasz-Lukyanenko-Tyson constructed horizontal maps from S^k to $\mathbb{H}^n$ which factor through n-spheres and showed that these maps have no smooth horizontal fillings. In this paper, however, we build on an example of Kaufman to show that these maps sometimes have Lipschitz fillings. This shows that the Lipschitz and the smooth horizontal homotopy groups of a space may differ. Conversely, we show that any Lipschitz map $S^k \to \mathbb{H}^1$ factors through a tree and is thus Lipschitz null-homotopic if $k \geq 2$.

DeJarnette, Hajłasz, Lukyanenko, and Tyson recently initiated a study of smooth horizontal homotopy groups $\pi_k^H(X)$ and Lipschitz homotopy groups $\pi_k^{\mathrm{Lip}}(X)$ when X is a sub-Riemannian manifold [DHLT11]. By definition, $\pi_k^H(X)$ (and $\pi_k^{\mathrm{Lip}}(X)$) consist of classes of smooth horizontal (respectively Lipschitz) maps $S^k \to X$, where two maps lie in the same class if there is a homotopy $S^k \times [0,1] \to X$ between them which is also smooth horizontal (resp. Lipschitz).

The groups $\pi_k^H(X)$ and $\pi_k^{\mathrm{Lip}}(X)$ capture more of the geometry of sub-Riemannian manifolds than the usual homotopy groups $\pi_k(X)$. For example, if $X = \mathbb{H}^n$ is the nth Heisenberg group with its standard Carnot-Caratéodory metric, it is homeomorphic to $\mathbb{R}^{2n+1}$, so its homotopy groups $\pi_k(\mathbb{H}^n)$ are trivial. Lipschitz maps to $\mathbb{H}^n$, however, are more complicated. If $f : D^k \to \mathbb{H}^n$ is Lipschitz, it must be a.e. Pansu differentiable [Pan89]. In particular, the rank of Df is a.e. at most n. Ambrosio-Kirchheim [AK00b] and Magnani [Mag04] showed that, as a consequence, if $k > n$, then $\mathcal{H}^k_{cc}(f(D^k)) = 0$. Therefore, if $\alpha : S^n \to \mathbb{H}^n$ is a smooth horizontal (and thus Lipschitz) embedding, it cannot be extended to a Lipschitz map of a ball [Gro96, BF09, RW10], so $\pi_n^H(\mathbb{H}^n)$ and $\pi_n^{\mathrm{Lip}}(\mathbb{H}^n)$ are non-trivial. In fact, these groups are uncountably generated [DHLT11].

The behavior of $\pi_k^H(\mathbb{H}^n)$ and $\pi_k^{\mathrm{Lip}}(\mathbb{H}^n)$ when $k > n$ is unknown. DeJarnette, Hajłasz, Lukyanenko, and Tyson [DHLT11] showed that if $\beta \in \pi_k(S^n)$ is nontrivial, then $\alpha \circ \beta : S^k \to \mathbb{H}^n$ is a nontrivial element of $\pi_k^H(\mathbb{H}^n)$ (their theorem is stated for a particular smooth embedding α, but their methods generalize to arbitrary smooth embeddings). Their proof relies on Sard's theorem, however, so it does not generalize to Lipschitz maps. They asked:

Question 1 ([DHLT11, 4.6])**.** *Is the map $\pi_k^H(\mathbb{H}^n) \to \pi_k^{Lip}(\mathbb{H}^n)$ an isomorphism?*

Question 2 ([DHLT11, 4.17])**.** *If $\alpha : S^n \to \mathbb{H}^n$ is a bilipschitz embedding, is the induced map $\pi_k^{Lip}(S^n) \to \pi_k^{Lip}(\mathbb{H}^n)$ an injection?*

The second author was supported by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada.

In this paper, we will show that even if $\beta \in \pi_k(S^n)$ is nontrivial, $\alpha \circ \beta$ may be Lipschitz-null homotopic, answering both of these questions in the negative. More precisely, we prove the following theorems.

Theorem 1. *If $\alpha : S^n \to \mathbb{H}^n$ and $\beta : S^k \to S^n$ are Lipschitz maps and $n + 2 \leq k < 2n - 1$, then $\alpha \circ \beta$ can be extended to a Lipschitz map $D^{k+1} \to \mathbb{H}^n$.*

Since this extension is Lipschitz, it is almost everywhere Pansu differentiable, and the Pansu differential has rank $\leq n$ wherever it is defined. Another version of our construction proves:

Theorem 2. *If $n + 1 \leq k < 2n - 1$, then any Lipschitz map $\beta : S^k \to S^n$ can be extended to a Lipschitz map $D^{k+1} \to \mathbb{R}^{n+1}$ whose derivative has rank $\leq n$ almost everywhere.*

Our constructions build on Kaufman's construction of a Lipschitz surjection from the unit cube to the unit square whose derivative has rank 1 almost everywhere [Kau79].

In view of the results above it is natural to ask whether $\pi_k^{\mathrm{Lip}}(\mathbb{H}^n)$ is trivial when $n+2 \leq k < 2n-1$. This may be hard to answer, since general Lipschitz k-spheres in $\mathbb{H}^n$ may be more complicated. While the spheres we consider in Theorems 1 and 2 have image with Hausdorff dimension n, the methods we use to prove the theorems can be adapted to produce Lipschitz maps of k-spheres to $\mathbb{H}^n$ whose image has Hausdorff dimension arbitrarily close to k.

When $n = 1$, however, things are much simpler. We will show:

Theorem 3. *If $k \geq 2$, then any Lipschitz map $f : S^k \to \mathbb{H}^1$ factors through a metric tree. That is, there is a metric tree Z and there are Lipschitz maps $\psi : S^k \to Z$ and $\varphi : Z \to \mathbb{H}^1$ such that $f = \varphi \circ \psi$.*

Recall that a metric tree or $\mathbb{R}$-tree is a geodesic metric space such that every geodesic triangle is isometric to a tripod. Note that these trees may still have large images; Kaufman's construction [Kau79], for instance, can be adapted to produce a Lipschitz map $S^5 \to \mathbb{H}^1$ whose image contains a ball in $\mathbb{H}^1$. As a consequence of Theorem 3 we obtain:

Corollary 4. *If $k \geq 2$ and $\alpha : S^k \to \mathbb{H}^1$, then α is Lipschitz null-homotopic. Furthermore, for any $\epsilon > 0$, α is ϵ-close to a map whose image has Hausdorff dimension 1.*

Proof. For the first statement, since Z is a metric tree, it is contractible by a Lipschitz homotopy $h : Z \times [0, 1] \to Z$. Composing this with ψ and φ gives a Lipschitz homotopy contracting α to a point.

For the second statement, let $\lambda = \mathrm{Lip}(f)$ and let E be a finite ϵ/λ net of points in S^k. Let T be the convex hull of $\psi(E)$ in Z; this is a finite tree. The closest-point projection $p : Z \to T$ is Lipschitz and moves each point a distance at most ϵ, so $\varphi \circ p \circ \psi$ is a Lipschitz map which is ϵ-close to α. Its image is $\varphi(T)$, which has Hausdorff dimension 1. □

Consequently, $\pi_k^{\mathrm{Lip}}(\mathbb{H}^1) = \{0\}$ for all $k \geq 2$. In general, a Lipschitz map $\alpha : X \to \mathbb{H}^n$ need not be ϵ-close to a map whose image has Hausdorff dimension n; the homotopies constructed in Theorem 1 cannot be approximated by such maps.

Theorem 3 is a special case of the following theorem. Recall that a metric space (X, d) is said to be quasi-convex if there exists C such that any two points $x, x' \in X$

can be joined by a curve of length at most $Cd(x, x')$. Furthermore, a metric space (Y, d) is called purely k-unrectifiable if $\mathcal{H}^k(\varrho(C)) = 0$ for every Lipschitz map ϱ from a Borel subset $C \subset \mathbb{R}^k$ to Y. It was shown in [AK00b, Mag04] that the Heisenberg group $\mathbb{H}^n$, endowed with a Carnot-Carathéodory metric, is purely k-unrectifiable for $k \geq n + 1$.

Theorem 5. *Let X be a quasi-convex metric space with $\pi_1^{Lip}(X) = 0$. Let furthermore Y be a purely 2-unrectifiable metric space. Then every Lipschitz map from X to Y factors through a metric tree.*

Theorem 5 will be proved in Section 3. If C is the quasi-convexity constant and ψ and φ are as above, then ψ can be chosen to be $C \operatorname{Lip}(f)$-Lipschitz and φ to be 1-Lipschitz. As a corollary, we find that $\pi_k^{\mathrm{Lip}}(Y) = \{0\}$ for all $k \geq 2$ and every purely 2-unrectifiable space Y.

1. Preliminaries

In this section we briefly collect some of the basic definitions and properties of the Heisenberg groups. We furthermore recall the necessary definitions of metric derivatives and currents in metric spaces which will be needed for the proof of Theorem 5.

1.1. **Heisenberg groups.** The nth Heisenberg group $\mathbb{H}^n$, where $n \geq 1$, is the Lie group given by $\mathbb{H}^n := \mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the group multiplication

$$(x, y, z) \odot (x', y', z') = (x + x', y + y', z + z' + \langle y, x' \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. A basis of left invariant vector fields on $\mathbb{H}^n$ is defined by

$$X_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, n,$$

and $Z = \frac{\partial}{\partial z}$. The subbundle $H\mathbb{H}^n \subset T\mathbb{H}^n$ generated by the vector fields $X_j, Y_j, j = 1, \ldots, n$, is called the horizontal subbundle. A C^1-smooth map $f : M \to \mathbb{H}^n$, where M is a smooth manifold, is called horizontal if the derivative df of f maps TM to the horizontal sub bundle $H\mathbb{H}^n$.

The Heisenberg group $\mathbb{H}^n$ is naturally equipped with a family $(s_r)_{r>0}$ of dilation homomorphisms $s_r : \mathbb{H}^n \to \mathbb{H}^n$ defined by

$$s_r(x, y, z) := (rx, ry, r^2 z).$$

Let g_0 be the left-invariant Riemannian metric on $\mathbb{H}^n$ such that the X_j, Y_k, Z are pointwise orthonormal. The Carnot-Carathéodory metric on $\mathbb{H}^n$ corresponding to g_0 is defined by

$$d(x, y) := \inf\{\text{length}_{g_0}(c) : \ c \text{ is a horizontal } C^1 \text{ curve from } x \text{ to } y\},$$

where $\text{length}_{g_0}(c)$ denotes the length of c with respect to g_0. The metric d on $\mathbb{H}^n$ is 1-homogeneous with respect to the dilations s_r, that is,

$$d(s_r(w), s_r(w')) = rd(w, w')$$

for all $w, w' \in \mathbb{H}^n$. Throughout this paper, $\mathbb{H}^n$ will always be equipped with the Carnot-Carathéodory metric d defined above or any metric which is biLipschitz equivalent to d.

1.2. **Metric derivatives.** We recall the definition of the metric derivative of a Lipschitz map from a Euclidean to a metric space, as introduced and studied by Kirchheim in [Kir94]. For this, let (X, d) be a metric space and $f : U \to X$ a Lipschitz map, where $U \subset \mathbb{R}^n$ is open. The metric derivative of f at $x \in U$ in direction $v \in \mathbb{R}^n$ is defined by

$$\operatorname{md} f_x(v) := \lim_{r \to 0^+} \frac{d(f(x + rv), f(x))}{r}$$

if the limit exists. It was shown in [Kir94] that for almost every $x \in U$ the metric derivative $\operatorname{md} f_x(v)$ exists for all $v \in \mathbb{R}^n$ and defines a semi-norm on $\mathbb{R}^n$. It can be shown that for any Lipschitz curve $c : [a, b] \to X$ we have

$$\operatorname{length}(c) = \int_a^b \operatorname{md} c_t(1) dt.$$

1.3. **Currents in metric spaces.** We recall some notions from the theory of metric currents developed by Ambrosio-Kirchheim in [AK00a]. For this, let (X, d) be a complete metric space and $m \geq 0$ an integer. Denote by $\operatorname{Lip}(X)$ and $\operatorname{Lip}_b(X)$ the spaces of Lipschitz, respectively, bounded Lipschitz functions on X. Currents generalize the idea of chains. A metric m-current in X is a multilinear functional $T : \operatorname{Lip}_b(X) \times [\operatorname{Lip}(X)]^m \to \mathbb{R}$ satisfying certain conditions; see [AK00a, Def. 3.1]. Evaluating a current on the tuple $(f, \pi_1, \dots, \pi_m)$ is roughly analogous to integrating the m-form $f \, d\pi_1 \wedge \cdots \wedge d\pi_m$ over a chain.

Currents satisfy continuity and finite mass conditions. The continuity condition asks that if π_i^k are Lipschitz functions on X with $\sup_{k,i} \operatorname{Lip}(\pi_i^k) < \infty$ and such that $\pi_i^k \to \pi_i$ pointwise as $k \to \infty$ then $T(f, \pi_1^k, \dots, \pi_m^k)$ converges to $T(f, \pi_1, \dots, \pi_m)$ as $k \to \infty$. The finite mass condition asks that that there exists a finite Borel measure μ on X such that

$$|T(f, \pi_1, \dots, \pi_m)| \leq \int_X |f| d\mu$$

for all $(f, \pi_1, \dots, \pi_m) \in \operatorname{Lip}_b(X) \times [\operatorname{Lip}(X)]^m$ with $\operatorname{Lip}(\pi_i) \leq 1$. The smallest such measure is denoted by $\|T\|$; the mass of T is the number $\mathbf{M}(T) := \|T\|(X)$.

If $m \geq 1$ then the boundary of an m-current T is the functional defined by

$$\partial T(f, \pi_1, \dots, \pi_{m-1}) := T(1, f, \pi_1, \dots, \pi_{m-1}).$$

If $\varphi : X \to Y$ is a Lipschitz map into a complete metric space Y then the pushforward of T by φ is defined by

$$\varphi_\# T(f, \pi_1, \dots, \pi_m) := T(f \circ \varphi, \pi_1 \circ \varphi, \dots, \pi_m \circ \varphi);$$

this defines a metric m-current in Y. For $\theta \in L^1(\mathbb{R}^m)$ a metric m-current in $\mathbb{R}^m$ is defined by

$$[\![\theta]\!](f, \pi_1, \dots, \pi_m) := \int_{\mathbb{R}^m} \theta f \det\left(\frac{\partial \pi_i}{\partial x_j}\right) d\mathcal{H}^m$$

for all $(f, \pi_1, \dots, \pi_m) \in \operatorname{Lip}_b(\mathbb{R}^m) \times [\operatorname{Lip}(\mathbb{R}^m)]^m$. In the sequel we will often write $[\![K]\!]$ instead of $[\![1_K]\!]$. We will furthermore denote by $[\![S^1]\!]$ the 1-current on $\mathbb{R}^2$ given by

$$[\![S^1]\!](f, \tau) = \int_0^{2\pi} f(e^{it}) \frac{d}{dt} \tau(e^{it}) dt$$

for all $(f,\tau) \in \mathrm{Lip}_b(\mathbb{R}^2) \times \mathrm{Lip}(\mathbb{R}^2)$. One has $\partial[\![D^2]\!] = [\![S^1]\!]$. If $\varphi : D^2 \to X$ is Lipschitz and such that $\mathrm{md}\,\varphi$ is degenerate almost everywhere then it is not difficult to check that $\varphi_\#[\![D^2]\!] = 0$.

2. Constructing extensions

In this section we prove Theorems 1 and 2. The restriction in these theorems that $k < 2n-1$ can be weakened somewhat. To state the theorems in full generality, we will need to recall some facts about the homotopy groups of wedges of spheres.

If $X = S^n \vee \cdots \vee S^n$ is an m-fold wedge of n-spheres, let $\iota_i : S^n \to X$, $i = 1, \dots, m$, be the map into the mth factor of X. In what follows, addition will be taken in $\pi_k(S^n)$ or $\pi_k(X)$, so $(\sum_{i=1}^m \iota_i)$ represents a sphere which wraps once around each factor of the wedge product.

If $\beta : S^k \to S^n$, then $\left(\sum_{i=1}^m \iota_i\right) \circ \beta$ and $\sum_{i=1}^m (\iota_i \circ \beta)$ are not homotopic in general. The Hilton-Milnor theorem describes the difference between these two maps:

Theorem 6 (cf. [Whi78, Thm. 8.3])**.** *If $m, n \geq 2$, there is an isomorphism*

$$\pi_k(X) \cong \underbrace{\pi_k(S^n) \oplus \cdots \oplus \pi_k(S^n)}_{m\ times} \oplus \bigoplus_{j=0}^{\infty} \pi_k(S^{q_j(n-1)+1}),$$

with q_j a sequence of integers going to ∞ with $q_j \geq 2$.

There are homomorphisms $h_j : \pi_k(S^n) \to \pi_k(S^{q_j(n-1)+1})$, $j = 0, 1, 2, \dots$, such that if $\beta \in \pi_k(S^n)$, then

$$\left(\sum_{i=1}^m \iota_i\right) \circ \beta = \sum_{i=1}^m (\iota_i \circ \beta) + \sum_{j=0}^{\infty} w_j \circ h_j(\beta)$$

where

$$w_j : S^{q_j(n-1)+1} \to X$$

is an iterated Whitehead product of the ι_i's with q_j terms.

When $m = 2$, the $h_j(\beta)$'s are known as the *Hopf-Hilton invariants* of β, and if the Hopf-Hilton invariants of β vanish, then

$$(*) \qquad \left(\sum_{i=1}^m \iota_i\right) \circ \beta = \sum_{i=1}^m (\iota_i \circ \beta) \text{ for all } q.$$

We can then generalize Theorems 1 and 2 as follows:

Theorem 7. *Let $n \geq 2$ and $n + 2 \leq k$. Let $\alpha : S^n \to \mathbb{H}^n$ be a Lipschitz map and let $\beta : S^k \to S^n$ be a Lipschitz map such that $(*)$ holds. Then $\alpha \circ \beta : S^k \to \mathbb{H}^n$ can be extended to a Lipschitz map $r : D^{k+1} \to \mathbb{H}^n$.*

Theorem 8. *Let $n \geq 2$ and $n + 1 \leq k$. Let $\beta : S^k \to S^n$ be a Lipschitz map such that $(*)$ holds. Then β can be extended to a Lipschitz map $\gamma : D^{k+1} \to \mathbb{R}^{n+1}$ whose derivative has rank $\leq n$ a.e.*

If $n \geq 2$ and $k < 2n - 1$, then $(*)$ holds, since each Hopf-Hilton invariant lies in $\pi_k(S^d)$ for some $d > k$. Theorems 1 and 2 thus follow from Theorems 7 and 8.

Equation $(*)$ also holds in other dimensions. For instance, when n is odd, $n \geq 3$, and $k = 2n - 1$, then $\pi_{2n-1}(S^n)$ is a finite group by Serre's Finiteness Theorem [Ser53]. The only h_i with a nontrivial target is h_0, which sends $\pi_{2n-1}(S^n)$ to

$\pi_{2n-1}(S^{2n-1}) = \mathbb{Z}$, but any homomorphism from a finite group to $\mathbb{Z}$ is trivial. Similar arguments hold, for instance, for $\beta \in \pi_9(S^3)$ or $\beta \in \pi_{11}(S^4)$.

The proof of Theorem 7 is based on that of Theorem 8, so we will prove it first.

Proof of Thm. 8. Let $I^n = [0,1]^n$ be the unit n-cube. It suffices to consider the case that $\beta : \partial I^{k+1} \to \partial I^{n+1}$ and construct an extension of β to all of I^{k+1}.

Our construction is based on a construction of Kaufman [Kau79]. We will construct a map on a cube by defining a Lipschitz map h on a cube with holes in it, then filling each of the holes with a scaling of h. Repeating this process defines a Lipschitz map on all of the cube except a Cantor set of measure zero, so we finish by extending the map to the Cantor set by continuity.

Let $\epsilon > 0$ be such that $(2\epsilon)^{-(k+1)} > \epsilon^{-(n+1)}$ and $\frac{1}{\epsilon} \in \mathbb{N}$. Subdivide I^{n+1} into a grid of $N = \epsilon^{-(n+1)}$ cubes of side length ϵ and let J be the n-skeleton of this grid. Number the subcubes $1, 2, \dots, N$ and let J_i be the ith subcube.

Subdivide I^{k+1} into $(2\epsilon)^{-(k+1)}$ cubes of side length 2ϵ and choose N of these subcubes, numbered $1, \dots, N$. For $i = 1, \dots, N$, we let K_i be a cube of side length ϵ, centered at the center of the ith subcube and let

$$K = I^{k+1} \setminus \bigcup_{i=1}^{N} K_i.$$

We will define a Lipschitz map $h : K \to J$ that sends the boundaries of cubes in K to the boundaries of cubes in J. Since the image is an n-complex in $\mathbb{R}^{n+1}$, the derivatives of h will have rank $\leq n$ a.e. First, we define h on ∂K. The boundary of K is $\partial I^{k+1} \cup \bigcup \partial K_i$; let $h = \beta$ on ∂I^{k+1}, and define h on ∂K_i as a scaling and translation β_i of β which sends ∂K_i to ∂J_i. So far, this definition is Lipschitz.

Next, we extend h. Choose basepoints $x \in \partial I^{k+1}$ and $x_i \in \partial K_i$ and a collection of non-intersecting curves λ_i connecting x to x_i. We can give K the structure of a CW-complex, with vertices $x, x_1, \dots, x_N$; edges λ_i; k-cells $\partial I^{k+1}, \partial K_1, \dots, \partial K_N$; and a single $(k+1)$-cell. We have already defined h on all of the vertices and k-cells, and since J is connected, we can extend h to the edges of K. It only remains to extend it to the $(k+1)$-cell.

Consider the map $g : S^k \to J$ coming from the boundary of the $(k+1)$-cell. The complex J is homotopy equivalent to $\vee^N S^n$, and we can choose the equivalence so that the inclusions into each factor correspond to maps $\iota_i : \partial I^{n+1} \to \partial J_i$, $i = 1, \dots, N$ into the boundary of each subcube. Then the inclusion $\iota : \partial I^{n+1} \hookrightarrow J$ of the boundary of the entire unit cube is homotopic to $\sum_{i=1}^N \iota_i$, and we have

$$\begin{aligned} g &= \iota \circ \beta - \sum_{i=1}^{N} \beta_i \\ &= \left(\sum_{i=1}^{N} \iota_i\right) \circ \beta - \sum_{i=1}^{N} (\iota_i \circ \beta) \end{aligned}$$

where the above equation is taken in $\pi_k(J)$. By hypothesis, this is null-homotopic, so h can be extended continuously to all of K. In fact, by a smoothing argument, this extension can be made Lipschitz.

We can construct an increasing sequence

$$X_0 = K \subset X_1 \subset X_2 \subset \dots$$

by gluing together scaled copies of K as follows. Let $X_0 = K$. To construct X_{i+1} from X_i, we glue a copy of K, scaled by ϵ^{i+1}, to each of the cubical holes of X_i. This replaces a hole of side length ϵ^{i+1} by N holes of side length ϵ^{i+2}, so for each i, X_i is the complement of N^{i+1} cubes of side length ϵ^{i+1} in I^{n+1}. The union $\bigcup_{i=0}^{\infty} X_i$ is the complement in I^{n+1} of a Cantor set of measure zero.

For each i, we will construct a map $r_i : X_i \to I^{n+1}$ such that,

- r_{i+1} extends r_i,
- $\operatorname{Lip} r_i \leq \operatorname{Lip} h$,
- the derivative of r_i has rank $\leq n$ a.e., and
- the restriction of r_i to the boundary of one of the holes of X_i is a copy of β scaled by ϵ^{i+1}.

Let $r_0 = h$ on X_0. This satisfies all the above conditions. For any i, we construct X_{i+1} from X_i by gluing copies of K to holes in X_i. On each new copy of K, we let r_{i+1} be a copy of h scaled by ϵ^{i+1}. This agrees with r_i on the boundary of the copy of K, and since we scaled the domain and the range by the same factor, we still have $\operatorname{Lip} r_{i+1} \leq \operatorname{Lip} h$.

The direct limit of the r_i is a map

$$r : \bigcup_{i=0}^{\infty} X_i \to I^{n+1}$$

defined on the complement of a Cantor set in I^{k+1} with $\operatorname{Lip} r \leq \operatorname{Lip} h$. If we extend r to all of I^{k+1} by continuity, we get a Lipschitz extension of β whose derivative has rank $\leq n$ a.e. □

The construction in the Heisenberg group is similar. Note that because fillings of n-spheres in the Heisenberg group have Hausdorff dimension $n+2$, we need $k \geq n+2$ rather than $k \geq n+1$. We will also need the following theorem about low-dimensional Lipschitz extensions to Heisenberg groups:

Theorem 9 ([Gro96], [WY10])**.** *For any n, there is a $c > 0$ such that if X is a cube complex of dimension $\leq n$, and if $f_0 : X^{(0)} \to \mathbb{H}^n$ is a Lipschitz map defined on the vertices of X, then there is a Lipschitz extension $f : X \to \mathbb{H}^n$ of f_0 such that* $\operatorname{Lip} f \leq c \operatorname{Lip} f_0$.

Proof of Thm. 7. As before, we may replace S^k and S^n with ∂I^{k+1} and ∂I^{n+1}. We will start by constructing an n-complex J which is homotopy equivalent to a wedge of spheres, a subset K of I^{k+1}, and a Lipschitz map $h : K \to J$. The main difference between this construction and the previous one is that J will be a complex equipped with a map $\bar{\alpha} : J \to \mathbb{H}^n$ rather than a subset of $\mathbb{H}^n$.

For any $\epsilon > 0$ such that $1/\epsilon \in \mathbb{N}$, consider the complex $((\partial I^{n+1}) \times [0, 1/\epsilon]) \cup (I^{n+1} \times \{0\})$. This has $2n+2$ faces of the form $I^n \times [0, 1/\epsilon]$ and one of the form I^{n+1} and we can tile it with a total of

$$N(\epsilon) = (2n+2)\epsilon^{-(n+2)} + \epsilon^{-(n+1)}$$

cubes of side length ϵ. Let $J(\epsilon)$ be the n-skeleton of this tiling. We identify ∂I^{n+1} with $\partial I^{n+1} \times \{1/\epsilon\} \subset J(\epsilon)$.

We claim that there is some $c > 0$ such that if $f : \partial I^{n+1} \to \mathbb{H}^n$ is a Lipschitz map, then for any $\epsilon > 0$, there is a Lipschitz extension $\bar{f} : J(\epsilon) \to \mathbb{H}^n$ with Lipschitz constant $\operatorname{Lip} \bar{f} \leq c \operatorname{Lip} f$. Recall that there is a family of dilations $s_t : \mathbb{H}^n \to \mathbb{H}^n$ such that $s_t(0) = 0$ for all t and $d(s_t(u), s_t(v)) = t d(u, v)$. After composing f with

a dilation of $\mathbb{H}^n$ and translating it so that its image contains the identity, we may assume that $\mathrm{Lip}(f) = 1$ and that $f(\partial I^{n+1})$ is contained in the ball $B \subset \mathbb{H}^n$ around the identity of radius $n+1$. Define $\bar{f} : J(\epsilon) \to \mathbb{H}^n$ on the vertices of $J(\epsilon)$ as

$$\bar{f}(v,t) = \begin{cases} s_{\epsilon t}(f(v)) & t > 0 \\ 1 & t = 0 \end{cases}$$

where $v \in I^{n+1}, t \in [0, 1/\epsilon]$ and where $s_t : \mathbb{H}^n \to \mathbb{H}^n$ is dilation by a factor of t. We claim that this is Lipschitz on the vertices with Lipschitz constant independent of ϵ; then, by Theorem 9, we can extend it to a c-Lipschitz map on all of $J(\epsilon)$.

It suffices to show that the distance between the images of any two adjacent vertices is $O(\epsilon)$, with implicit constant depending only on n. If the two vertices are $(v, 0)$ and $(v', 0)$, the map sends both of them to the identity. If v is adjacent to v' in ∂I^{n+1} and $t \in (0, 1/\epsilon]$, then

$$d(\bar{f}(v,t), \bar{f}(v',t)) = d(s_{\epsilon t}(f(v)), s_{\epsilon t}(f(v'))) \le d(v, v') = \epsilon.$$

If the vertices are of the form $(v,t), (v,t')$, with $|t - t'| = \epsilon$, let $f(v) = (x, y, z)$ for $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. On any compact set,

$$d_{\mathbb{H}^n}((a,b,c),(a',b',c')) = O(\sqrt{\|a-a'\| + \|b-b'\| + \|c-c'\|}),$$

and since $\bar{f}(v,t), \bar{f}(v,t') \in B$,

$$d_{\mathbb{H}^n}(\bar{f}(v,t), \bar{f}(v,t')) = O(\sqrt{\epsilon|t-t'|\|x\| + \epsilon|t-t'|\|y\| + \epsilon^2|t^2-t'^2|\|z\|}) = O(\epsilon)$$

as desired.

Choose $\epsilon > 0$ such that

$$N(\epsilon) \le \left\lfloor \frac{1}{2c\epsilon} \right\rfloor^{k+1}$$

(this is possible because $k \ge n+2$) and let $J = J(\epsilon)$, $N = N(\epsilon)$. Label the cubes of J by $1, \dots, N$.

Next, we construct K. We can subdivide I^{k+1} into at least N subcubes, each with side length at least $2c\epsilon$. Number N of these subcubes $1, \dots, N$, and for each i, let K_i be a cube of side length $c\epsilon$ centered at the center of the ith subcube. Let $K = I^{k+1} \smallsetminus \bigcup_{i=1}^N K_i$. As in the proof of Prop. 8, construct a Lipschitz map $h : K \to J$ such that for each i, ∂K_i is mapped to ∂J_i by a scaling of β.

Define $X_0 = K \subset X_1 \subset \dots$ as before, so that X_i consists of I^{k+1} with N^{i+1} cubical holes of side length $(c\epsilon)^{i+1}$. Let $Y_0 = J$. This consists of N cubical holes of side length ϵ. For each i, we let Y_{i+1} be Y_i with a scaled copy of J glued to each cubical hole, so that for each i, Y_i is an n-complex consisting of the boundaries of N^{i+1} cubes of side length ϵ^{i+1}. We construct maps $\gamma_i : X_i \to Y_i$ inductively. We start by letting $\gamma_0 = h$. By induction, if C is the boundary of one of the holes in X_i, γ_i sends C to the boundary D of a hole in Y_i. To construct X_{i+1} from X_i, we glue a scaled copy of K to C, and to construct Y_{i+1} from Y_i, we glue a scaled copy of J to D. We extend γ_i to γ_{i+1} by sending each scaled copy of K to the corresponding scaled copy of J by a scaled copy of h. Note that since the scaling factors in the construction of X_i and Y_i are different, the Lipschitz constant of γ_i varies from point to point; if Z is a connected component of $X_i \setminus X_{i-1}$, then

$$\mathrm{Lip}\, \gamma_i|_Z \le c^{-i}\, \mathrm{Lip}\, h.$$

Finally, we construct maps $\sigma_i : Y_i \to \mathbb{H}^n$. We proceed inductively. As noted above, any Lipschitz map $f : \partial I^{n+1} \to \mathbb{H}^n$ can be extended to a Lipschitz map

$\bar{f} : J \to \mathbb{H}^n$ with $\operatorname{Lip} \bar{f} \le c \operatorname{Lip} f$. We will construct a sequence of maps $\sigma_i : Y_i \to \mathbb{H}^n$ with $\operatorname{Lip} \sigma_i \le c^{i+1} \operatorname{Lip} \alpha$. Let $\sigma_0 = \bar{\alpha} : Y_0 \to \mathbb{H}^n$; we have $\operatorname{Lip} \sigma_0 \le c \operatorname{Lip} \alpha$. For each i, the complex Y_{i+1} consists of Y_i with N^{i+1} copies of J glued on, so we can extend σ_i to Y_{i+1} by constructing an extension over every copy of J. By induction, $\operatorname{Lip} \sigma_i \le c^{i+1} \operatorname{Lip} \alpha$, so $\operatorname{Lip} \sigma_{i+1} \le c^{i+2} \operatorname{Lip} \alpha$ as desired.

Let $r_i = \sigma_i \circ \gamma_i$. If Z is a connected component of $X_i \setminus X_{i-1}$, then

$$\operatorname{Lip} r_i|_Z \le c^{-i} (\operatorname{Lip} h) c^{i+1} (\operatorname{Lip} \alpha) \le c (\operatorname{Lip} h)(\operatorname{Lip} \alpha),$$

so the r_i are uniformly Lipschitz. Their direct limit is a Lipschitz map from the complement of a Cantor set to $\mathbb{H}^n$ which extends $\alpha \circ \beta$. Extending this to all of I^{k+1} by continuity, we get the desired r. □

3. Factoring through trees

The aim of this section is to prove Theorem 5. For this, let X and Y be metric spaces as in the statement of the theorem and let $f : X \to Y$ be a Lipschitz map. Roughly, the idea of the proof is to pull back the metric of Y by f and show that the resulting metric space is a tree.

Define a pseudo-metric on X by

$$d_f(x, x') := \inf\{\text{length}(f \circ c) : \ c \text{ Lipschitz curve from } x \text{ to } x'\}$$

and note that

$$d_f(x, x') \le C \operatorname{Lip}(f) d(x, x') \tag{1}$$

for all $x, x' \in X$, where C is the quasi-convexity constant of X. Let Z be the quotient space $Z := X/_\sim$ by the equivalence relation given by $x \sim x'$ if and only if $d_f(x, x') = 0$. Endow Z with the metric

$$d_Z([x], [x']) := d_f(x, x'),$$

where $[x]$ denotes the equivalence class of x, and define maps $\psi : X \to Z$ and $\varphi : Z \to Y$ by $\psi(x) := [x]$ and $\varphi([x]) := f(x)$. It follows from (1) that ψ is $C \operatorname{Lip}(f)$-Lipschitz. Since

$$d(f(x), f(x')) \le \text{length}(f \circ c)$$

for all Lipschitz curves c from x to x' we moreover infer that φ is well-defined and 1-Lipschitz. Note that $f = \varphi \circ \psi$.

It remains to show that Z is a metric tree. By [Wen08, Proposition 3.1] it is enough to prove that Z is a geodesic metric space and that every Lipschitz curve $\alpha : S^1 \to Z$ is trivial when viewed as a metric 1-current; that is, $\alpha_\# [\![S^1]\!] = 0$ in the notation of Section 1.3. For this, let $\alpha : S^1 \to Z$ be a Lipschitz curve. It is not difficult to see that there exist Lipschitz curves $\beta_n : S^1 \to X$, $n = 1, 2, \dots$, such that $\psi \circ \beta_n$ converges uniformly to α and such that $\operatorname{Lip}(\psi \circ \beta_n) \le 2 \operatorname{Lip}(\alpha)$ for every $n \in \mathbb{N}$. It follows from the continuity property of currents that $(\psi \circ \beta_n)_\# [\![S^1]\!]$ converges weakly to $\alpha_\# [\![S^1]\!]$. In order to prove that $\alpha_\# [\![S^1]\!] = 0$ it therefore suffices to prove that $(\psi \circ \beta)_\# [\![S^1]\!] = 0$ for every Lipschitz curve $\beta : S^1 \to X$.

Let $\beta : S^1 \to X$ be a Lipschitz curve. By the hypotheses on X, there is some Lipschitz map $\varrho : D^2 \to X$ which extends β. We claim that the metric derivative $\operatorname{md}(\psi \circ \varrho)_z$ is degenerate for almost every $z \in D^2$. From this it then follows that $(\psi \circ \varrho)_\# [\![D^2]\!] = 0$ and hence

$$(\psi \circ \beta)_\# [\![S^1]\!] = \partial (\psi \circ \varrho)_\# [\![D^2]\!] = 0.$$

In order to prove that $\operatorname{md}(\psi\circ\varrho)$ is degenerate almost everywhere, let $0<\varepsilon<1/2$ and let $\{v_1,\dots,v_k\}\subset S^1$ be a finite $\frac{\varepsilon}{\lambda}$-dense subset, where $\lambda=\operatorname{Lip}(f\circ\varrho)$. For $i\in\{1,2,\dots,k\}$ define

$$A_i := \{z\in D^2 :\ \operatorname{md}(f\circ\varrho)_z \text{ exists, is a seminorm, and } \operatorname{md}(f\circ\varrho)_z(v_i)\leq\varepsilon\}.$$

It is not difficult to show that

$$\left|D^2\backslash\bigcup_{i=1}^k A_i\right| = 0, \tag{2}$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^2$. Indeed, for almost every $z\in D^2$ the metric derivative $\operatorname{md}(f\circ\varrho)_z$ exists and is a seminorm. Since Y is purely 2-unrectifiable it follows from the area formula that $\operatorname{md}(f\circ\varrho)_z$ is degenerate for almost every $z\in D^2$. Thus, given such z, there exists $v\in S^1$ such that $\operatorname{md}(f\circ\varrho)_z(v)=0$. Choose i such that $|v-v_i|\leq\varepsilon/\lambda$. It follows that

$$\operatorname{md}(f\circ\varrho)_z(v_i)\leq\operatorname{md}(f\circ\varrho)_z(v_i-v)\leq\varepsilon.$$

This proves (2). Now, fix $i\in\{1,2,\dots,k\}$ and let $z\in A_i$ be a Lebesgue density point. Let $r_0>0$ be such that $B(z,2r_0)\subset D^2$ and

$$\frac{|B(z,r)\backslash A_i|}{|B(z,r)|}\leq 100^{-1}\varepsilon^2 \tag{3}$$

for all $r\in(0,2r_0)$. Let $v_i^\perp\in S^1$ be a vector orthogonal to v_i and let $r\in(0,r_0)$. For each $s\in(0,\varepsilon r)$ let C_s denote the set $C_s:=\{t\in[0,r]: z+sv_i^\perp+tv_i\notin A_i\}$. It follows from Fubini's theorem and (3) that there exists a subset $\Omega\subset(0,\varepsilon r)$ of strictly positive measure such that $\mathcal{H}^1(C_s)\leq\varepsilon r$ for every $s\in\Omega$. Let $s\in\Omega$ and denote by γ the piecewise affine curve in $\mathbb{R}^2$ connecting z with $z+rv_i$ via $z+sv_i^\perp$ and $z+sv_i^\perp+rv_i$. It now follows that

$$\operatorname{length}(f\circ\varrho\circ\gamma)\leq 2s\lambda+\int_0^r\operatorname{md}(f\circ\varrho)_{z+sv_i^\perp+tv_i}(v_i)dt\leq 2s\lambda+\varepsilon r+\lambda|C_s|\leq(3\lambda+1)\varepsilon r$$

and hence that for every $r\in(0,r_0)$

$$\frac{1}{r}d_Z(\psi\circ\varrho(z),\psi\circ\varrho(z+rv_i))=\frac{1}{r}d_f(\varrho(z),\varrho(z+rv_i))\leq(3\lambda+1)\varepsilon.$$

In particular, if $\operatorname{md}(\psi\circ\varrho)$ exists at z and is a seminorm then $\operatorname{md}(\psi\circ\varrho)_z(v_i)\leq(3\lambda+1)\varepsilon$. Since $\varepsilon>0$ was arbitrary this shows that $\operatorname{md}(\psi\circ\varrho)_z$ is degenerate for almost all $z\in D^2$. It follows that $(\psi\circ\varrho)_\#[\![D^2]\!]=0$ and thus also $(\psi\circ\beta)_\#[\![S^1]\!]=0$ as claimed.

It remains to show that Z is geodesic. Clearly, Z is a length space. Let $z,z'\in Z$ and let $\alpha_n:[0,1]\to Z$ be a sequence of Lipschitz curves from z to z' such that $\operatorname{length}(\alpha_n)\to d_Z(z,z')$. Define $T_n:=\alpha_{n\#}[\![[0,1]]\!]$ and note that $\mathbf{M}(T_n)\leq\operatorname{length}(\alpha_n)$ and, by the above, $T_n=T_1$ for all n. It thus follows that $\mathbf{M}(T_1)\leq d_Z(z,z')$. By [RW12, Lemma 3.12] there exists a Lipschitz curve $\alpha:[0,1]\to Z$ such that $\partial(\alpha_\#[\![[0,1]]\!])=\partial T_1$ and $\operatorname{length}(\alpha)\leq\mathbf{M}(T_1)$; hence α connects z and z' and has length $d_Z(z,z')$. This shows that Z is geodesic and thus, by [Wen08, Proposition 3.1], that Z is a metric tree. This concludes the proof of Theorem 5.

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